ENGINEERING METHODS OF DETERMINING THERMAL BOUNDARY CONDITIONS BY MEANS OF TEMPERATURE MEASUREMENT DATA

V. I. Zhuk and A. S. Golosov

UDC 536.24.02

Engineering methods of solving the inverse heat-conduction problem, approved in the practice of processing data of nonstationary thermal experiments, are elucidated.

The determination of boundary conditions (BC) by means of the data of temperature measurements is carried out during processing and analyzing the results of experimental investigations. Direct measurement of the intensity of the thermal action in an experiment is difficult, as a rule, and usually the nonstationary temperature field of the characteristic elements in the object under investigation is recorded. The BC characteristics, the specific heat fluxes or the heat-exchange coefficients governing the thermal mode of the test, must be sought for by means of available information about heating of the object during analysis of the test data. The problem reduces to finding the solution of the inverse problem for the nonstationary heat-conduction equation (IPHC). In general, formulation of the inverse heat-conduction problem is incorrect in the sense that there is no continuous dependence of the results of the solution on the input data, which is especially essential for applied problems when the input information (data of the experiment) contains errors in measurement and decoding.

Because of its timeliness, such a problem has been examined by a number of authors. A survey of the main papers is presented in [1, 2, 3]. The first publication of results is in the papers of Kudryavtsev, Chakalev, and Shumakov [4-7]. According to our information, A. M. Zhuravskii proposed the first solutions of certain problems in 1954.

The state of the question under consideration is such that, as a rule, each investigator starts from the requirements of practice and independently develops his own method of processing the temperature measurement data taking account of the singularities of the measurement facilities used and the specifics of the processes being investigated. At the same time the degree of development of the theory and the practical methods at this instant permits carrying out an objective analysis of the state of the art and of noting the optimal means of solving typical practical problems.

Let us consider the solution of the IPHC for nonstationary, relatively short-range, intensive thermal modes of an experiment.

The use of any method of solving the IPHC is determined by the structural peculiarities and the location of the temperature sensor in the object under investigation. In practice, it is convenient to use imbedded sensors, thermocouples in special casings. In the simplest case, the temperature of the heated surface of the structure is measured successfully, while in other cases temperature measurements are possible only deep in the walls at some distance from the heating surface and, in a number of cases, only on a surface opposite to that being heated. For nonmetal structures with a possible entrainment of the material of the surface being heated, the BC can apparently be determined by measuring the temperature just in the bulk of the structure.

In conformity with the measurement methods considered and taking into account the specifics of the processes of heat exchange and heating of structure, it is recommended that a method of determining the thermal BC be constructed by means of the temperature measurement data:

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 1, pp. 45-50, July, 1975. Original article submitted January 17, 1975.

Simple algorithms to convert the BC should be worked out in the case of measuring the temperature of the surface being heated or near it;

Economical algorithms should be used for IPHC of general form, at least because of the introduction of definite constraints by means of a priori estimates of possible assumptions (linearization, approximation of the laws of time variation of the heat-exchange characteristics, the motion of boundaries, etc.);

Algorithms with minimal constraints should be worked out to process unique experimental data.

Algorithms to convert the BC according to the results of temperature measurements of the surface being heated [1, 8] are convenient for the practical purposes of processing the data of thermal experiments. This problem is correct [2].

The relationships [2]

$$q(t) = \frac{\lambda}{\sqrt{\pi a}}\left[\frac{T_w(t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{T_w(t) - T_w(\tau)}{(t-\tau)^{3/2}}\,d\tau\right] \tag{1}$$

or [1, 8]

$$q(t) = \frac{\lambda}{\sqrt{\pi a}}\int_0^t \frac{\frac{dT_w(\tau)}{d\tau}}{\sqrt{t-\tau}}\,d\tau \tag{2}$$

can be used for the case of a semi-infinite body. Separating the range of computation [0, t] into k sufficiently small intervals Δt and denoting the discrete values of the measured surface temperature by

$$T_w(s\Delta t) = T_w(t_s) = T_s,\ s = 1,\ 2, \ldots,\ k, \tag{3}$$

we approximately determine the integral in the right side of (2):

$$\int_0^t \frac{T_w'(\tau)}{\sqrt{t-\tau}}\,d\tau \approx \sum_{s=1}^{k}\int_{(s-1)\Delta t}^{s\Delta t}\frac{T_w'(\tau)}{\sqrt{t_k-\tau}}\,d\tau \approx \frac{2}{\sqrt{\Delta t}}\,\frac{T_s - T_{s-1}}{\sqrt{k-s+1}+\sqrt{k-s}},$$

so that

$$q(k\Delta t) = \frac{\lambda}{\sqrt{\pi a\Delta t}}\sum_{s=1}^{k}(T_s - T_{s-1})\,C_{k-s}, \tag{4}$$

where

$$C_m = 2(\sqrt{m+1} - \sqrt{m}). \tag{5}$$

Formula (4) is acceptable upon compliance with the condition

$$\mathrm{Fo} \ll 1, \tag{6}$$

and in practice for

$$\mathrm{Fo} \leqslant 0.1. \tag{6'}$$

It is especially convenient for operational processing of temperature measurement data "by hand" on keyboard machines or on small electronic computers.

Analogous dependences can be recommended for finite-thickness plates also [1, 8],

$$q(k\Delta t) = 2\,\frac{\lambda}{R}\sum_{s=1}^{k}(T_s - T_{s-1})\,C_{k-s}(\Delta\mathrm{Fo}), \tag{7}$$

where the coefficients

$$C_m(\Delta\mathrm{Fo}) = \sum_{n=1}^{\infty}\frac{1}{\mu_n^2\Delta\mathrm{Fo}}\left\{\exp\left[-\mu_n^2 m\Delta\mathrm{Fo}\right] - \exp\left[-\mu_n^2(m+1)\Delta\mathrm{Fo}\right]\right\} \tag{8}$$

have been tabulated for a number of values of m and ΔFo or can be evaluated by means of a special subprogram during the computations of (5) on an electronic computer.

Appropriate computational dependences to determine the heat flux to a heated surface by means of its measured temperature have been obtained also for infinitely long solid and hollow cylinders.

For the solid cylinder

$$q(k\Delta t) = 2\frac{\lambda}{R}\sum_{s=1}^{k}(T_s - T_{s-1})B_{k-s}(\Delta\text{Fo}), \tag{9}$$

$$B_m(\Delta\text{Fo}) = \sum_{n=1}^{\infty}\frac{1}{\nu_n^2\Delta\text{Fo}}\{\exp[-\nu_n^2 m\Delta\text{Fo}] - \exp[-\nu_n^2(m+1)\Delta\text{Fo}]\}. \tag{10}$$

Tables have also been compiled for the coefficients $B_m(\Delta\text{Fo})$. In the case of a hollow cylinder, unilateral heating from within or without is considered when the other surface is considered heat insulated [15].

The problem of converting the BC in a nonlinear formulation can be realized by solving the direct problem by some difference method and then calculating the heat fluxes by means of the temperature gradient at the surface [1].

To process the data of temperature measurements by deep-lying sensors, when it is necessary to take account of the incorrectness of the IPHC, direct methods are acceptable in a number of practical cases [1, 9, 10, 11]. The following recursion relations can hence be used:

semi-infinite body,

$$q(k\Delta t) = \frac{\frac{\lambda}{X}(T_k - T_0) - \sum_{s=1}^{k-1} q_s C_{k-s}(\Delta\text{Fo}_X)}{C_0(\Delta\text{Fo}_X)}, \tag{11}$$

$$C_m(\Delta\text{Fo}_X) = 2\sqrt{\Delta\text{Fo}_X}\left[\sqrt{m+1}\,\text{ierfc}\frac{1}{2\sqrt{(m+1)\Delta\text{Fo}_X}} - \sqrt{m}\,\text{ierfc}\frac{1}{2\sqrt{m\Delta\text{Fo}_X}}\right]; \tag{12}$$

infinite plate [1],

$$q(k\Delta t) = \frac{\frac{c\rho R}{\Delta t}(T_k - T_0) - \sum_{s=1}^{k} q_s[1 - 2C_{k-s}(\Delta\text{Fo}, \xi)]}{1 - 2C_0(\Delta\text{Fo}, \xi)}, \tag{13}$$

$$C_m(\Delta\text{Fo}, \xi) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos\alpha_n\xi}{\alpha_n^2\Delta\text{Fo}}\{\exp[-\alpha_n^2 m\Delta\text{Fo}] - \exp[-\alpha_n^2(m+1)\Delta\text{Fo}]\}; \tag{14}$$

solid cylinder,

$$q(k\Delta t) = \frac{\frac{c\rho R}{2\Delta t}(T_k - T_0) - \sum_{s=1}^{k-1} q_s[1 + B_{k-s}(\Delta\text{Fo}, \eta)]}{1 + B_0(\Delta\text{Fo}, \eta)}, \tag{15}$$

$$B_m(\Delta\text{Fo}, \eta) = \sum_{n=1}^{\infty}\frac{J_0(\beta_n\eta)}{\beta_n^2 J_0(\beta_n)\Delta\text{Fo}}\{\exp[-\beta_n^2 m\Delta\text{Fo}] - \exp[-\beta_n^2(m+1)\Delta\text{Fo}]\}; \tag{16}$$

hollow cylinder heated from within,

$$q(k\Delta t) = \frac{\frac{\lambda(1-p^2)}{2R_1\Delta\text{Fo}}(T_k - T_0) - \sum_{s=1}^{k-1} q_s[1 - B_{k-s}(\Delta\text{Fo}, \theta, p)]}{1 - B_0(\Delta\text{Fo}, \theta, p)} \tag{17}$$

$$B_m(\Delta\text{Fo}, \theta, p) = -\pi\frac{1-p^2}{2p\Delta\text{Fo}} \times$$

$$\times\sum_{n=1}^{\infty}\frac{J_1^2(\gamma_n)}{\gamma_n[J_1^2(p\gamma_n) - J_1^2(\gamma_n)]}[J_0(\theta\gamma_n)Y_1(p\gamma_n) - Y_0(\theta\gamma_n)J_1(p\gamma_n)] \times$$

$$\times\{\exp[-\gamma_n^2 m\Delta\text{Fo}] - \exp[-\gamma_n^2(m+1)\Delta\text{Fo}]\}; \tag{18}$$

hollow cylinder heated from without,

$$q(k\Delta t) = \frac{\dfrac{\lambda(1-p^2)}{2R_2\Delta\mathrm{Fo}}(T_k - T_0) - \sum_{s=1}^{k-1} q_s[1 - D_{k-s}(\Delta\mathrm{Fo},\ \theta,\ p)]}{1 - D_0(\Delta\mathrm{Fo},\ \theta,\ p)}, \tag{19}$$

$$D_m(\Delta\mathrm{Fo},\ \theta,\ p) = \pi\frac{1-p^2}{2\Delta\mathrm{Fo}}\sum_{n=1}^{\infty}\frac{J_1(p\gamma_n)J_1(\gamma_n)}{\gamma_n[J_1^2(p\gamma_n) - J_1^2(\gamma_n)]}$$

$$\times[J_0(\theta\gamma_n)Y_1(p\gamma_n) - Y_0(\theta\gamma_n)J_1(p\gamma_n)]\{\exp[-\gamma_n^2 m\Delta\mathrm{Fo}] - \exp[-\gamma_n^2(m+1)\Delta\mathrm{Fo}]\}. \tag{20}$$

According to the results in [9], the computational formulas proposed above are suitable upon compliance with the condition

$$\Delta\mathrm{Fo}_X \geqslant 0.31. \tag{21}$$

On the basis of our computational experience, the more rigorous constraint

$$\Delta\mathrm{Fo}_X \geqslant 0.4\text{—}0.45 \tag{21'}$$

should be used.

The domain of application of computational formulas of the type (11), (13), (15), (17), (19) is broadened because of raising their stability by the method of least squares [9]. Thus, in determining heat fluxes in the k-th computational interval by this method by using values of the temperature measured at the points k + j, the stability limit of the formulas is raised to $\Delta\mathrm{Fo}_X \geq 0.012$ for j = 3. The computational formula hence has the following form (infinite plate):

$$q(k\Delta t) = \frac{\sum_{i=0}^{j}\left[\dfrac{c\rho R}{\Delta t}(T_{k+j} - T_0) - \sum_{s=1}^{k-1} q_s A_{k-s+j}\right]\sum_{i=0}^{j} A_i}{\sum_{s=0}^{j}\left(\sum_{i=0}^{s} A_i\right)^2}, \tag{22}$$

$$A_i = 1 - 2C_i(\Delta\mathrm{Fo},\ \xi). \tag{23}$$

On the basis of the computational schemes proposed, algorithms have been developed and programs to compute the heat fluxes by means of the temperature measured on structural elements of simple geometric shape by using coefficients tabulated or calculated in advance (or during the solution of the IPHC) have been devised.

Regularization methods [12, 13, 14], by means of which appropriate algorithms and standard programs must be compiled, should be used for data-processing problems for a thermal experiment for whose solution the above-mentioned constraints on the spacing are not acceptable, and also for nonlinear problems.

NOTATION

t, time; x, coordinate; X, spacing between the heated surface and the point with the measured temperature; T, measured temperature; λ, coefficient of thermal conductivity; c, specific heat; ρ, density; a, coefficient of thermal diffusivity; q, specific heat flux; Δt, computation time interval; R, plate thickness or outer radius of the solid cylinder; R_1, inner radius of the hollow cylinder; R_2, outer radius of the hollow cylinder; $\mathrm{Fo} = at/l^2$; $\Delta\mathrm{Fo} = a\Delta t/l^2$; l, characteristic geometric dimension, equal to R for a plate or R_2 for a solid cylinder; $\Delta\mathrm{Fo}_X = a\Delta t/X^2$; $\xi = X/R$; $\eta = r/R$; $\theta = r/R_2$; $\mu_n = (2n-1)\pi/2$; ν_n, roots of the characteristic equation $J_0(\nu_n) = 0$; $\alpha_n = n\Pi$; γ_n, roots of the characteristic equation $J_1(p\gamma_n)Y_1(\gamma_n) - Y_1(p\gamma_n)J_1(\gamma_n) = 0$; $p = R_1R_2$; β_n roots of the characteristic equation $J_1(\beta_n) = 0$; $Y_0(z)$, $Y_1(z)$, zero- and first- order Bessel functions of the first kind; $J_0(z)$, $J_1(z)$, zero- and first-order Bessel functions of the second kind; 0, 1, 2, ..., k refer to running times; k refers to computational time; n refers to the ordinal number of the roots of the characteristic equation.

LITERATURE CITED

1. G. T. Aldoshin, A. S. Golosov, and V. I. Zhuk, Heat and Mass Transfer [in Russian], Vol. 8, Nauka i Tekhnika, Minsk (1968).
2. O. M. Alifanov, Inzh.-Fiz. Zh., 25, No. 3 (1973).
3. A. G. Temkin, Inverse Methods of Heat Conduction [in Russian], Énergiya, Moscow (1973).
4. E. V. Kudryavtsev and N. V. Shumakov, Zh. Tekh. Fiz., 27, No. 4 (1957).
5. N. V. Shumakov, Zh. Tekh. Fiz., 27, No. 4 (1957).
6. K. N. Chakalev and N. V. Shumakov, Inzh.-Fiz. Zh., 1, No. 3 (1958).
7. E. V. Kudryavtsev, K. N. Chakalev, and N. V. Shumakov, Nonstationary Heat Exchange [in Russian], Izd. Akad. Nauk SSSR, Moscow (1961).
8. V. I. Zhuk and A. S. Golosov, Heat and Mass Transfer in Solids, Liquids, and Gases [in Russian], ITMO, Minsk (1970).
9. J. V. Beck, ASME Paper No. HT-46 (1962).
10. J. V. Beck and H. Wolf, ASME Paper, No. 65-HT-40 (1965).
11. O. M. Alifanov, Inzh.-Fiz. Zh., 24, No. 6 (1973).
12. O. M. Alifanov, Inzh.-Fiz. Zh., 25, No. 2 (1973).
13. O. M. Alifanov, Inzh.-Fiz. Zh., 23, No. 6 (1972).
14. O. M. Alifanov, Inzh.-Fiz. Zh., 24, No. 2 (1973).
15. A. A. Koshkin, in: Works of the Central Institite of Aviation Machine Construction [in Russian], No. 35 (1972).